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Inequalities for Continuous-Spin Ising Ferromagnets

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We investigate correlation inequalities for Ising ferromagnets with continuous spins, giving a simple unified derivation of inequalities of Griffiths, Ginibre, Percus, Lebowitz, and Ellis and Monroe. The single-spin measure and Hamiltonian for which an inequality may be proved become more restricted as the inequality becomes more complex. However, all results hold for a model with ferromagnetic pair interactions, positive (nonuniform) external field, and single-spin measure ν either $\nu(\sigma) = [1/(l + 1)] \times \sum_{j=0}^{l} \delta(-l + 2j + \sigma)$ (spin $l/2$) or $d\nu(\sigma) = \exp[-P(\sigma)]\, d\sigma$, where P is an even polynomial all of whose coefficients must be positive except the quadratic, which is arbitrary. The Lebowitz correlation inequality is a corollary of the Ellis–Monroe inequality. As an application, we generalize the method of van Beijeren to establish a sharp phase interface at low temperature in nearest neighbor ferromagnets of at least three dimensions with arbitrary (symmetric) single-spin measure.

KEY WORDS: Ising model; ferromagnetic; correlation inequality.

1. INTRODUCTION

In this paper we investigate correlation inequalities for ferromagnetic Ising models with continuous spins. These continuous-spin models, which we describe fully at the close of the introduction, generalize the classical spin-$\frac{1}{2}$ Ising models in that the spin variable is not restricted to the two values ± 1 but instead may assume any real value with some a priori single-spin weighting measure. Such Ising models can be used to represent many physical situations.

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In particular, they rigorously approximate scalar quantum field theories in the Euclidean region (lattice approximation[20]), and inequalities proved for continuous-spin models yield analogous inequalities for the more singular models of scalar quantum fields.

In Section 2 we give a simple unified derivation of many correlation inequalities for continuous-spin Ising ferromagnets, obtaining them for a large class of single-spin weighting measures. These inequalities were established in various special cases by Griffiths,[9] Ginibre,[7] Percus,[19] Ellis and Monroe,[4] Lebowitz,[17] and Griffiths *et al.*[13]. Although the single-spin measure and Hamiltonian for which an inequality may be proved become more restricted as the inequality becomes more complex, all inequalities hold for a model with ferromagnetic pair interactions, positive (nonuniform) external field, and single-spin measure either

$$d\nu(\sigma) = \frac{1}{l+1} \sum_{j=0}^{l} \delta(-l + 2j + \sigma) \quad (\text{spin } l/2)$$

or $d\nu(\sigma) = \exp[-P(\sigma)]\, d\sigma$. Here P is an even polynomial all of whose coefficients must be positive, except the quadratic, which is arbitrary. (Recent work by Ellis and Newman[6] elegantly relaxes this condition on P: it need only be an even differentiable function whose derivative is convex on $[0, \infty)$.) We exhibit interrelationships among these inequalities, deriving the Lebowitz inequality from the Ellis–Monroe inequality in the same way the second Griffiths inequality may be derived from the Ginibre inequality. The Griffiths–Hurst–Sherman inequality for concavity of magnetization is a corollary of the Lebowitz correlation inequality, as is an inequality which at zero external field shows the fourth Ursell function u_4 is negative.

In Section 3 we briefly discuss the need for the restrictions in the hypotheses of the theorems proved in the previous section. We conclude by combining the methods and results of Section 2 with the general theorem on phase transitions in Refs. 2 and 22 to extend the work of van Beijeren[1] on sharp phase separation to the case of arbitrary (symmetric) single-spin measure.

This paper is somewhat in the nature of a review, since to exhibit unity of method we have included proofs of some known results which fit naturally into our scheme. In accord with this partly expository character, but with no attempt at completeness, we now make some short historical remarks on the inequalities of Section 2 and mention a few of their applications. For a more thorough review, see Refs. 11, 12, 17, 20, 22, and references therein.

Theorem 2.1 and Corollary 2.1, the Griffiths inequalities, were obtained (in a slightly weaker form) by Griffiths[9] for spin-$\frac{1}{2}$ spins, pair interactions, and positive external field. They were strengthened and generalized to the case of polynomial interactions by Kelly and Sherman[14] and extended to higher discrete spins and certain continuous spins by Griffiths.[10] Theorem 2.2 is due to Ginibre,[7] with Theorem 2.1 and Corollary 2.1 as corollaries.

Well-known[11,12,17,20,22] applications of the Griffiths inequalities include the construction and shape independence of the infinite-volume limit of spin expectations, monotone increase of the spin expectations in the couplings of the Hamiltonian, and persistence of phase transitions when couplings are added or increased.

Theorem 2.3 was proved for spin-$\frac{1}{2}$ models by Percus.[19] Corollary 2.3, the Lebowitz correlation inequality, was obtained for spin-$\frac{1}{2}$ models by Lebowitz,[15] with the G.H.S. inequality (Corollary 2.4) as a special case. It was in a paper devoted to this inequality that Ellis and Monroe[4] established Theorem 2.4 for spin-$\frac{1}{2}$ spins by means of Gaussian random variables, though it appeared in a somewhat different guise. The proof was simplified and extended to continuous spins {having single-spin measure $\exp[-P(\sigma)]\,d\sigma$, P even, with all coefficients but the quadratic positive} independently by Ellis (Ref. 3 and later work, Refs. 5 and 6) and the present author. Derivation of the Lebowitz correlation inequality (Corollary 2.3) as a corollary is new here.

Among the many applications of these results, we mention that the G.H.S. inequality yields concavity of magnetization,[13] monotone decrease of the correlation length in the external field,[16] absence of a phase transition except at zero external field for models with pair Hamiltonians,[13] and inequalities among the lowest three eigenvalues of the anharmonic oscillator.[6,20] The Lebowitz correlation inequality, which is closely related to the Gaussian inequality of Newman,[18] has been used to bound expectations and correlations of many spins in terms of expectations and correlations of pairs of spins.[8] Such bounds may be used, for example, to simplify the construction of the infinite-volume limit,[8,22] and in quantum field theory, to show that the ϕ_2^4 theory has no even bound states.[21]

We conclude our introductory remarks with a formal definition of a finite, continuous-spin ferromagnetic Ising model. A finite, continuous-spin Ising ferromagnet is a triple (Λ, H, ν), where:

1. The set of sites Λ is a finite set. We associate with each site $i \in \Lambda$ a real spin variable $\sigma_i \in \mathbb{R}$; the product $\prod_{i\in\Lambda} \mathbb{R}$ is called the configuration space.

2. The Hamiltonian H is a polynomial on the configuration space, and the ferromagnetism assumption is that H has nonpositive coefficients. We write

$$H(\sigma) = -\sum_{K\in F_0(\Lambda)} J_K \sigma_K, \qquad J_K \geqslant 0 \tag{1}$$

where the coefficients J_K are called couplings (or bonds), $F_0(\Lambda)$ is the set of finite families ("sets" with repeated elements) in Λ, and σ_K is by definition the product

$$\sigma_K = \prod_{i \in K} \sigma_i$$

3. The single-spin measure ν is an even Borel probability measure on $\mathbb{R}$ which decays sufficiently rapidly that if $d = \deg(H)$ is the degree of the polynomial H,

$$\int_{\mathbb{R}} \exp(a|\sigma|^d)\, d\nu(\sigma) < \infty \qquad \forall a \in \mathbb{R} \tag{2}$$

The linear term $-\sum_{i\in\Lambda} J_i\sigma_i$ in H is commonly thought of as describing the effect of an external magnetic field, while higher order terms are considered to arise from the mutual interactions of the spins. We usually recognize this distinction by writing $-\sum_{i\in\Lambda} h_i\sigma_i$ in the Hamiltonian in place of $-\sum_{i\in\Lambda} J_i\sigma_i$. A pair interaction is a Hamiltonian of degree two.

The Gibbs measure μ of (Λ, H, ν) is the probability measure on the configuration space $\prod_{i\in\Lambda} \mathbb{R}$ defined by

$$\mu(E) = Z^{-1} \int_E \exp[-\beta H(\sigma)] \prod_{i\in\Lambda} d\nu(\sigma_i), \qquad E \subset \textstyle\prod_\Lambda \mathbb{R} \text{ measurable} \tag{3}$$

Here $\beta = 1/kT \in [0, \infty)$ is the inverse temperature and Z the partition function

$$Z = \int_{\Pi_\Lambda \mathbb{R}} \exp[-\beta H(\sigma)] \prod_{i\in\Lambda} d\nu(\sigma_i) \tag{4}$$

We indicate (thermal) expectations with respect to the Gibbs measure at inverse temperature β by angular brackets $\langle\ ; H, \nu, \beta\rangle$, omitting the descriptive arguments H, ν, β when they are clear from context:

$$\langle f; H, \nu, \beta\rangle = \langle f\rangle = \int_{\Pi_\Lambda \mathbb{R}} f\, d\mu = Z^{-1} \int_{\Pi_\Lambda \mathbb{R}} f e^{-\beta H} \prod_\Lambda d\nu \tag{5}$$

Physically, the sites Λ may be interpreted as the positions of atoms in a crystal, and the spin variable σ_i at each site $i \in \Lambda$ as a classical version of the quantum mechanical spin associated with the atom at i. The single-spin measure is a temperature-independent weight determined by internal properties of the atoms. A point σ in the configuration space $\prod_\Lambda \mathbb{R}$ corresponds to a state of the system, and $H(\sigma)$ is the energy of that state. If we allow the crystal to exchange energy with a large heat bath at inverse temperature β, the equilibrium state will be described by the canonical ensemble. Roughly speaking, this means that the probability of finding the system in some subset $E \subset \prod_\Lambda \mathbb{R}$ of the configuration space is given by the Gibbs measure $\mu(E)$.

2. BASIC INEQUALITIES

We now state and prove the inequalities for ferromagnetic Ising models mentioned in Section 1. The proofs employ the method of duplicate variables. Consider a finite ferromagnetic Ising model (Λ, H, ν). It is convenient to take

$$\Lambda = \{1, 2,..., N\}$$

so that the spin variables are $\sigma_1, \sigma_2,..., \sigma_N$. Construct the doubled system $(\Lambda \oplus \Lambda, H \oplus H, \nu)$, where $\Lambda \oplus \Lambda$ is the disjoint union of two copies of Λ, the $2N$ spin variables are $\sigma_1, \sigma_2,..., \sigma_N, \tau_1, \tau_2,..., \tau_N$, and the Hamiltonian $H \oplus H$ is $H(\sigma_1,..., \sigma_N) + H(\tau_1,..., \tau_N)$. Thus, the doubled system consists of two copies of the original model that do not interact with each other. Define the transformed variables

$$t_i = (1/\sqrt{2})(\sigma_i + \tau_i), \qquad q_i = (1/\sqrt{2})(\sigma_i - \tau_i), \qquad i \in \Lambda \tag{6}$$

where the $1/\sqrt{2}$ factors merely serve to make the transformation orthogonal. Construct also a redoubled system $(\Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda, H \oplus H \oplus H \oplus H, \nu)$ composed of four noninteracting copies of the original, with spins $\sigma_1,..., \sigma_N$, $\tau_1,..., \tau_N$, $\sigma_1',..., \sigma_N'$, $\tau_1',..., \tau_N'$, and the Hamiltonian $H(\sigma_1,..., \sigma_N) + H(\tau_1,..., \tau_N) + H(\sigma_1',..., \sigma_N') + H(\tau_1',..., \tau_N')$. As before, define

$$\begin{aligned} t_i &= (1/\sqrt{2})(\sigma_i + \tau_i), & q_i &= (1/\sqrt{2})(\sigma_i - \tau_i) \\ t_i' &= (1/\sqrt{2})(\sigma_i' + \tau_i'), & q_i' &= (1/\sqrt{2})(\sigma_i' - \tau_i'), \qquad i \in \Lambda \end{aligned} \tag{7}$$

Now set

$$\begin{aligned} \alpha_i &= (1/\sqrt{2})(t_i + t_i'), & \beta_i &= (1/\sqrt{2})(t_i - t_i') \\ \gamma_i &= (1/\sqrt{2})(q_i' + q_i), & \delta_i &= (1/\sqrt{2})(q_i' - q_i), \qquad i \in \Lambda \end{aligned} \tag{8}$$

Note the reversal of primes between α, β and γ, δ.

With this notation we have the following theorems:

Theorem 2.1 (First Griffiths Inequality). Let $A \in F_0(\Lambda)$ be a family of sites in a finite Ising ferromagnet (Λ, H, ν) with Hamiltonian

$$H = -\sum_{K \in F_0(\Lambda)} J_K \sigma_K, \qquad J_K \geqslant 0$$

and arbitrary (even) single-spin measure ν. Then

$$\langle \sigma_A \rangle \geqslant 0 \tag{9}$$

Theorem 2.2 (Ginibre Inequality). Let $A, B \in F_0(\Lambda)$ be families of sites in a finite Ising ferromagnet (Λ, H, ν) with Hamiltonian

$$H = -\sum_{K \in F_0(\Lambda)} J_K \sigma_K, \qquad J_K \geqslant 0$$

and arbitrary (even) single-spin measure ν. Then

$$\langle q_A t_B \rangle \geqslant 0 \tag{10}$$

where q, t are defined by (6).

Corollary 2.1 (Second Griffiths Inequality). Let $A,B \in F_0(\Lambda)$ be families of sites in the model of Theorem 2.2. Then

$$(\partial/\partial J_B)\langle \sigma_A \rangle \equiv \langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \geqslant 0 \tag{11}$$

Theorem 2.3 (Percus Inequality). Let $A \in F_0(\Lambda)$ be a family of sites in a finite Ising model (Λ, H, ν) with pair Hamiltonian

$$H = -\sum_{i \leqslant j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i, \qquad J_{ij} \geqslant 0 \quad \text{and} \quad h_i \textit{ arbitrary}$$

and arbitrary (even) single-spin measure ν. Then

$$\langle q_A \rangle \geqslant 0 \tag{12}$$

where q is defined by (6).

Corollary 2.2. Let i, j be sites in the model of Theorem 2.3. Then

$$(\partial/\partial h_j)\langle \sigma_i \rangle \equiv \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \geqslant 0 \tag{13}$$

Theorem 2.4 (Ellis–Monroe Inequality). Let $A,B,C,D \in F_0(\Lambda)$ be sites in a finite Ising ferromagnet (Λ, H, ν) with pair Hamiltonian

$$H = -\sum_{i \leqslant j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i, \qquad J_{ij} \geqslant 0, \quad h_i \geqslant 0$$

and single-spin measure ν either discrete and of the form

$$\nu(\sigma) = \frac{1}{l+1}\sum_{j=0}^{l} \delta(-l + 2j + \sigma) \qquad \left(\text{spin } \frac{l}{2}\right) \tag{14a}$$

or continuous and of the form

$$d\nu(\sigma) = \exp[-P(\sigma)]\, d\sigma \Big/ \int_{\mathbb{R}} \exp[-P(s)]\, ds \tag{14b}$$

where P is an even polynomial whose leading coefficient is positive, whose quadratic and constant coefficients are arbitrary, and whose remaining coefficients are nonnegative. Then

$$\langle \alpha_A \beta_B \gamma_C \delta_D \rangle \geqslant 0 \tag{15}$$

where α, β, γ, and δ are defined by (8).

Corollary 2.3 (Lebowitz Correlation Inequality). Let $A,B \in F_0(\Lambda)$ be families of sites in the model of Theorem 2.4. Then

$$\langle t_A t_B\rangle - \langle t_A\rangle\langle t_B\rangle \geqslant 0 \tag{16a}$$

$$\langle q_A q_B\rangle - \langle q_A\rangle\langle q_B\rangle \geqslant 0 \tag{16b}$$

$$\langle t_A q_B\rangle - \langle t_A\rangle\langle q_B\rangle \leqslant 0 \tag{16c}$$

where t, q are defined by (6).

Corollary 2.4 (Griffiths–Hurst–Sherman Inequality). Let i, j, k be sites in the model of Theorem 2.4. Then

$$(\partial^2/\partial h_j\, \partial h_k)\langle\sigma_i\rangle \equiv \langle\sigma_i\sigma_j\sigma_k\rangle - \langle\sigma_i\rangle\langle\sigma_j\sigma_k\rangle - \langle\sigma_j\rangle\langle\sigma_i\sigma_k\rangle - \langle\sigma_k\rangle\langle\sigma_i\sigma_j\rangle + 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle \leqslant 0 \tag{17}$$

Corollary 2.5. Let i, j, k, l be sites in the model of Theorem 2.4. Then

$$\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_l\rangle - \langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_l\rangle - \langle\sigma_i\sigma_l\rangle\langle\sigma_j\sigma_k\rangle + 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle\langle\sigma_l\rangle \leqslant 0 \tag{18}$$

The proofs of Theorems 2.1, 2.2, and 2.4 all proceed similarly, by reduction to the case of a model with a single site and zero Hamiltonian. The inverse temperature β is inessential and we set it equal to one. We must show that a thermal expectation

$$\langle f\rangle = \int fe^{-H}\prod d\nu \Big/ \int e^{-H}\prod d\nu$$

is nonnegative. The partition function in the denominator is positive, so we ignore it. We first verify that in the transformed variables, the Hamiltonian is a polynomial with nonpositive coefficients. Expanding e^{-H} in its Taylor series, we obtain a sum with nonnegative coefficients of integrals of products of the transformed variables against the product of the single-spin measures. Since each integral factors over the sites, it suffices to show that for a single site the integral of any product of the transformed variables is nonnegative; that is, that the theorem holds for one-site models with zero Hamiltonian. This is what we do. In the proof of Theorem 2.3 the reduction cannot proceed quite as far, but essentially the same method prevails. This reduction makes it clear that in all our results we could allow a different single-spin measure at each site, though such models are not commonly studied. Corollary 2.3 follows from Theorem 2.4 just as Corollary 2.1 follows from Theorem 2.2. Corollary 2.2 and Corollaries 2.4, and 2.5 are important special cases of Theorem 2.3 and Corollary 2.3, respectively.

Proofs

Theorem 2.1 (Proof). We want to show

$$\int_{\mathbb{R}^N} \sigma_A \exp\left(\sum_{K\in F_0(\Lambda)} J_K\sigma_K\right) d\nu(\sigma_1)\cdots d\nu(\sigma_N) \geqslant 0 \tag{19}$$

By expanding the exponential in its Taylor series and factoring the integrals over the sites as described in the previous paragraph, we reduce the problem to showing

$$\int_{\mathbb{R}} \sigma^n \, d\nu(\sigma) \geqslant 0 \qquad \forall n \tag{20}$$

By the symmetry of ν this vanishes when n is odd, and when n is even the integrand is nonnegative. *QED*

Theorem 2.2 (Proof). In terms of the transformed variables q and t the total Hamiltonian $H(\sigma) + H(\tau)$ is

$$-\sum_{K\in F_0(\Lambda)} J_K\left[\left(\frac{t+q}{\sqrt{2}}\right)_K + \left(\frac{t-q}{\sqrt{2}}\right)_K\right] \tag{21}$$

This is a polynomial in the t's and q's with nonpositive coefficients, because when we expand the product $\prod_{k\in K}(t_k - q_k)$ any negative term which appears is canceled by the corresponding term from the expansion of $\prod_{k\in K}(t_k + q_k)$. Now by expanding the exponential and factoring the integrals over the sites we reduce the problem to showing

$$\int_{\mathbb{R}^2} t^m q^n \, d\nu(\sigma)\, d\nu(\tau) \geqslant 0 \qquad \forall m, n \tag{22}$$

This vanishes by symmetry unless n and m are both even, in which case the integrand is nonnegative. *QED*

Theorem 2.3 (Proof). The transformation (6) is orthogonal, so in terms of the transformed variables q and t the Hamiltonian $H(\sigma) + H(\tau)$ is

$$-\sum_{i\leqslant j} J_{ij}(q_iq_j + t_it_j) - 2^{1/2}\sum_i h_it_i \tag{23}$$

We want to show

$$\int_{\mathbb{R}^2} q_A \exp\left(\sum_{i\leqslant j} J_{ij}q_iq_j\right) \exp\left(\sum_{i\leqslant j} J_{ij}t_it_j + 2^{1/2}\sum h_it_i\right)$$
$$\times\, d\nu(\sigma_1)\, d\nu(\tau_1)\cdots d\nu(\sigma_N)\, d\nu(\tau_N) \geqslant 0 \tag{24}$$

By expanding the first exponential $\exp(\sum_{i \leqslant j} J_{ij} q_i q_j)$, we see that it suffices to show

$$\int_{\mathbb{R}^N} \left(\prod_{k=1}^{N} [q_k]^{n_k} \right) \exp\left(\sum_{i \leqslant j} J_{ij} t_i t_j + 2^{1/2} \sum_i h_i t_i \right) d\nu(\sigma_1) \cdots d\nu(\tau_N) \geqslant 0 \qquad (25)$$

for all possible exponents n_k. But this integral vanishes by symmetry unless all the n_k are even, in which case the integrand is positive. *QED*

Theorem 2.4 (Proof). The transformation (8) is orthogonal, so in terms of the transformed variables α, β γ, and δ the total Hamiltonian $H(\sigma) + H(\tau) + H(\sigma') + H(\tau')$ is

$$-\sum_{i \leqslant j} J_{ij}(\alpha_i \alpha_j + \beta_i \beta_j + \gamma_i \gamma_j + \delta_i \delta_j) - 2 \sum_i h_i \alpha_i \qquad (26)$$

Since this is a polynomial with nonpositive coefficients, by expanding the exponential and factoring the integrals over the sites we reduce the problem to showing

$$\int_{\mathbb{R}^2} \alpha^k \beta^l \gamma^m \delta^n \, d\nu(\sigma) \, d\nu(\tau) \, d\nu(\sigma') \, d\nu(\tau') \geqslant 0 \qquad \forall k, l, m, n \qquad (27)$$

By symmetry (27) vanishes unless k, l, m, and n all have the same parity. When this parity is even the integrand is nonnegative, so we restrict our further attention to the case of odd parity. At this point we distinguish between discrete and continuous spins.

In the discrete case it suffices to consider spins of $\frac{1}{2}$,

$$\nu(\sigma) = \tfrac{1}{2}[\delta(\sigma + 1) + \delta(\sigma - 1)] \qquad (28)$$

for since our transformation of variables is linear the Griffiths "analog system" method[10] may be applied to generate the higher spin results from the spin-$\frac{1}{2}$ case. (The analog system method represents a higher spin by a sum of spins of $\frac{1}{2}$ in a suitably enlarged model.) Because the exponents k, l, m and n are all odd, we may factor out $\alpha\beta\gamma\delta$:

$$\alpha^k \beta^l \gamma^m \delta^n = (\alpha^{k-1} \beta^{l-1} \gamma^{m-1} \delta^{n-1}) \alpha\beta\gamma\delta \qquad (29)$$

The first factor is nonnegative since it has even exponents. The second factor is also nonnegative: since $\sigma^2 \doteq \tau^2 = \sigma'^2 = \tau'^2$ for spins of $\frac{1}{2}$, we find

$$\alpha\beta\gamma\delta = \tfrac{1}{4}(\sigma\tau - \sigma'\tau')^2 \geqslant 0 \qquad (30)$$

In the continuous case our problem is to show

$$\int_{\mathbb{R}^4} \alpha^k \beta^l \gamma^m \delta^n \exp[-P(\sigma) - P(\tau) - P(\sigma') - P(\tau')] \, d\sigma \, d\tau \, d\sigma' \, d\tau' \geqslant 0 \qquad (31)$$

for odd k, l, m, n. We claim that when $P(\sigma) + \cdots + P(\tau')$ is expressed in terms of α, β, γ, and δ it has the special form

$$P(\sigma) + \cdots + P(\tau') = Q(\alpha^2, \beta^2, \gamma^2, \delta^2) - \alpha\beta\gamma\delta R(\alpha^2, \beta^2, \gamma^2, \delta^2) \tag{32}$$

where Q and R are polynomials with nonnegative coefficients, except possibly for the coefficients of α^2, β^2, γ^2, and δ^2 in Q. Temporarily accepting this claim, and recalling that transformation (8) is orthogonal, we see that the integral (31) becomes

$$\int_{\mathbb{R}^4} \alpha^k \cdots \delta^n \exp[\alpha\beta\gamma\delta R(\alpha^2,\ldots, \delta^2) - Q(\alpha^2,\ldots, \delta^2)]\, d\alpha \cdots d\delta \tag{33}$$

Replacing α by $-\alpha$ and averaging gives

$$\int_{\mathbb{R}^4} (\alpha^{k-1}\beta^{l-1}\gamma^{m-1}\delta^{n-1})\{\alpha\beta\gamma\delta \sinh[\alpha\beta\gamma\delta R(\alpha^2,\ldots, \delta^2)]\} \times \exp[-Q(\alpha^2,\ldots, \delta^2)]\, d\alpha \cdots d\delta \tag{34}$$

The first factor, in parentheses, is nonnegative because it has even exponents; the second factor, in braces, is nonnegative because $R(\alpha^2,\ldots, \delta^2) \geqslant 0$; the third factor is obviously nonnegative.

It remains only to verify claim (32). It suffices to consider the case of a monomial, $P(X) = X^{2p}$. Expanding with the multinomial theorem gives

$$\begin{aligned}
&\sigma^{2p} + \tau^{2p} + \sigma'^{2p} + \tau'^{2p} \\
&\quad = \left(\frac{\alpha + \beta + \gamma - \delta}{2}\right)^{2p} + \left(\frac{\alpha + \beta - \gamma + \delta}{2}\right)^{2p} \\
&\qquad + \left(\frac{\alpha - \beta + \gamma + \delta}{2}\right)^{2p} + \left(\frac{\alpha - \beta - \gamma - \delta}{2}\right)^{2p} \\
&\quad = 2^{-2p} \sum_{a+b+c+d=2p} \frac{(2p)!}{a!\, b!\, c!\, d!} \\
&\qquad \times [(-1)^d + (-1)^c + (-1)^b + (-1)^{d+c+b}]\alpha^a\beta^b\gamma^c\delta^d
\end{aligned} \tag{35}$$

The coefficient of $\alpha^a\beta^b\gamma^c\delta^d$ vanishes unless a, b, c, and d all have the same parity; it is positive when this parity is even, and, it is negative when the parity is odd. This observation immediately yields claim (32). *QED*

Corollary 2.1 (Proof). We want to show

$$\langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle \geqslant 0$$

Using the doubled system, we find

$$\langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle = \langle\sigma_A\sigma_B - \sigma_A\tau_B\rangle = \left\langle\left(\frac{t+q}{\sqrt{2}}\right)_A\left[\left(\frac{t+q}{\sqrt{2}}\right)_B - \left(\frac{t-q}{\sqrt{2}}\right)_B\right]\right\rangle \tag{36}$$

This is the expectation of a polynomial in the q's and t's, which may be shown to have nonnegative coefficients just as (21) was shown to have nonpositive coefficients. By Theorem 2.2 this expectation is nonnegative. *QED*

Corollary 2.2 (Proof). Corollary 2.2 is a special case of Theorem 2.3:

$$0 \leqslant \langle q_i q_j\rangle = \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle \tag{37}$$

Corollary 2.3 (Proof). We want to show

$$\langle t_A t_B\rangle - \langle t_A\rangle\langle t_B\rangle \geqslant 0$$
$$\langle q_A q_B\rangle - \langle q_A\rangle\langle q_B\rangle \geqslant 0$$
$$\langle t_A\rangle\langle q_B\rangle - \langle t_A q_B\rangle \geqslant 0$$

Using the redoubled system, we have

$$\langle t_A t_B\rangle - \langle t_A\rangle\langle t_B\rangle$$
$$= \langle t_A t_B - t_A t_B'\rangle = \left\langle\left(\frac{\alpha+\beta}{\sqrt{2}}\right)_A\left[\left(\frac{\alpha+\beta}{\sqrt{2}}\right)_B - \left(\frac{\alpha-\beta}{\sqrt{2}}\right)_B\right]\right\rangle \tag{38a}$$

$$\langle q_A q_B\rangle - \langle q_A\rangle\langle q_B\rangle$$
$$= \langle q_A' q_B' - q_A' q_B\rangle = \left\langle\left(\frac{\gamma+\delta}{\sqrt{2}}\right)_A\left[\left(\frac{\gamma+\delta}{\sqrt{2}}\right)_B - \left(\frac{\gamma-\delta}{\sqrt{2}}\right)_B\right]\right\rangle \tag{38b}$$

$$\langle t_A\rangle\langle q_B\rangle - \langle t_A q_B\rangle$$
$$= \langle t_A q_B' - t_A q_B\rangle = \left\langle\left(\frac{\alpha+\beta}{\sqrt{2}}\right)_A\left[\left(\frac{\gamma+\delta}{\sqrt{2}}\right)_B - \left(\frac{\gamma-\delta}{\sqrt{2}}\right)_B\right]\right\rangle \tag{38c}$$

In each case the right-hand side is the expectation of a polynomial in α, β, γ, and δ with nonnegative coefficients. By Theorem 2.4, these expectations are nonnegative. *QED*

Corollary 2.4 (Proof). As noted by Lebowitz,[15] Corollary 2.4 is a special case of Corollary 2.3:

$$0 \geqslant \langle q_i q_j t_k\rangle - \langle q_i q_j\rangle\langle t_k\rangle = \frac{1}{\sqrt{2}}\frac{\partial^2}{\partial h_j\,\partial h_k}\langle\sigma_i\rangle \quad QED \tag{39}$$

Corollary 2.5 (Proof). Corollary 2.5 is obtained by symmetrizing the special case

$$\langle t_i t_j q_k q_l\rangle - \langle t_i t_j\rangle\langle q_k q_l\rangle \leqslant 0 \tag{40}$$

of Corollary 2.3. *QED*

It is pleasant to see that all these results essentially follow from symmetry arguments and the fact that the integral of a positive function is positive.

3. DISCUSSION

We begin this section by discussing the restrictions in the hypotheses of the theorems of Section 2. We point out some single-spin measures not covered by the hypotheses of all these theorems for which the conclusions follow by a limiting procedure, the most notable being the restriction of Lebesgue measure to some finite interval $[-b, b]$. Finally, as an application of the methods and results of Section 2, we outline the extension of the work of van Beijeren[(1)] on sharp phase interface to models with arbitrary (even) single-spin measure.

Let us turn now to the hypotheses of the theorems of the preceding section. As Theorems 2.1, 2.2, and Corollary 2.1 have been widely analyzed,[(7,9,14,19,20,22)] we shall not comment on them here.

The hypotheses of Theorem 2.3 and Corollary 2.2 are somewhat unusual in that the single-spin measure is arbitrary while the Hamiltonian is restricted to a pair interaction. To see that this restriction is valid, note that Corollary 2.2 fails for a spin-$\frac{1}{2}$ model with three sites $\{1, 2, 3\} = \Lambda$ and cubic Hamiltonian

$$H = -\sigma_1\sigma_2\sigma_3 + h\sigma_3, \qquad h > 0 \tag{41}$$

[We find that $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$ both vanish, but

$$\langle\sigma_1\sigma_2\rangle = -\tanh(1)\tanh(h) < 0$$

in contrast to (13).]

The hypotheses of Theorem 2.4 and its corollaries, Corollaries 2.3–2.5, contain restrictions both on the Hamiltonian and on the single-spin measure. Example 7.3 of Ref. 14 shows that the restriction of the Hamiltonian to pair interactions is needed. However, the constraint on the single-spin measure is more severe than required. To carry through the proof, we needed a certain polynomial $R(\alpha^2, \beta^2, \gamma^2, \delta^2)$ to be nonnegative. The hypotheses of the theorem ensured this by causing R to have positive coefficients, but $R \geqslant 0$ clearly follows from weaker assumptions. Theorem 2.4 is studied from this viewpoint in Appendix A of Ref. 22. Additionally, Ellis and Newman[(6)] have recently derived the elegant criterion that Theorem 2.4 and its corollaries hold provided P is an even function with a first derivative that is convex on $[0, \infty)$. It is also valid for single-spin measures obtained by a limiting process from those explicitly permitted. For example, Lebesgue measure restricted to the finite interval $[-b, b]$ is given as the limit

$$\frac{1}{2b}\chi_{[-b,b]}(\sigma)\,d\sigma = \lim_{n\to\infty}\frac{\exp[-(\sigma/b)^{2n}]\,d\sigma}{\int_{\mathbb{R}}\exp[-(s/b)^{2n}]\,ds} \tag{42}$$

of measures for which Theorem 2.4 holds. Note further that the spin-$\frac{1}{2}$ (Bernoulli) measure may be recovered from the continuous measures $\exp[-P(\sigma)]\,d\sigma$ as the limit

$$\lim_{q\to\infty} \{ \exp[-q(\sigma^2 - 1)^2]\,d\sigma \Big/ \int_{\mathbb{R}} \exp[-q(s^2 - 1)^2]\,ds \} \tag{43}$$

However, some constraint on the single-spin measure is necessary. For example, Corollary 2.5 fails for a one-site model with zero Hamiltonian having single-spin measure

$$a\delta(\sigma + 1) + (1 - 2a)\delta(\sigma) + a\delta(\sigma - 1), \qquad 0 < a < \tfrac{1}{6} \tag{44}$$

since

$$\langle\sigma^4\rangle - 3\langle\sigma^2\rangle^2 = 2a(1 - 6a) > 0 \tag{45}$$

in contrast to (18). It also fails for a one-site model having single-spin measure $\exp[-P(\sigma)]\,d\sigma$, where

$$P(\sigma) = q\sigma^2(\sigma - 1)^2(\sigma + 1)^2 + \sigma^2 \log\left(\frac{1}{2a} - 1\right) + \log\frac{1}{1 - 2a}, \qquad 0 < a < \frac{1}{6} \tag{46}$$

and q is sufficiently large,[20] because as $q \to \infty$, this measure converges to the preceding one.

We conclude by sketching the extension to models with arbitrary single-spin measure of the spin-$\frac{1}{2}$ results of van Beijeren[1] on sharp phase interface. (Full details of this generalization are given in Ref. 22.) Consider the nearest neighbor, isotropic, spin-$\frac{1}{2}$ Ising ferromagnet I_n in dimension $n \geqslant 3$ at zero external field, and take the inverse temperature β of the model larger than the critical inverse temperature $\beta_{c,n-1}$ of I_{n-1}. Van Beijeren shows that if a uniform magnetic field $h > 0$ is applied at the sites in the "top" half of the model, and an opposite field $(-h)$ at the sites in the "bottom" half of the model, then upon decreasing h to zero, we leave the model in an equilibrium state which has a sharp phase boundary: for some $c > 0$ the magnetization $\langle\sigma_i\rangle$ is at least c if site i is in the top half of the model and at most $-c$ in the bottom half. To establish this, one may combine the method of proof of Theorem 2.2 with Corollary 2.2 to show that the magnetization in the top half of the model is bounded below by the magnetization of the $(n - 1)$-dimensional nearest neighbor ferromagnet having the same physical parameters (temperature, coupling, and external field). By symmetry, the magnetization in the bottom half of the model is bounded above by the negative of the magnetization of this $(n - 1)$-dimensional slice. But for $n \geqslant 3$, I_{n-1} is spontaneously magnetized for $\beta > \beta_c$ ($\lim_{h\to 0}\langle\sigma_i; h\rangle > 0$), so this comparison implies that I_n has an equilibrium state with a sharp phase interface at sufficiently high reciprocal temperature.

In rough summary, the sharp phase boundary is a consequence of the method of Theorem 2.2, Corollary 2.2, and the spontaneous magnetization of I_{n-1}. As we have seen, Theorem 2.2 and Corollary 2.2 hold for models with arbitrary (even) single-spin measure. In Refs. 2 and 22 it is proved that nearest neighbor models in dimension at least two whose single-spin measure is not the δ-function are spontaneously magnetized at sufficiently large inverse temperature β. Combining these results yields:

Theorem 3.1. Let $(\mathbb{Z}^n, H, \nu)$ be the nearest neighbor Ising ferromagnet in dimension $n \geqslant 3$ with Hamiltonian

$$H = -J \sum_{i \in \mathbb{Z}^n} \sum_{\alpha=1}^{n} \sigma_i \sigma_{i+1_\alpha}, \qquad J > 0, \quad 1_\alpha = (\underbrace{0,\dots,0,1}_{\alpha},\underbrace{0,\dots,0}_{n-\alpha}), \tag{47}$$

and $\nu \neq \delta$. Let m_s be the spontaneous magnetization of the nearest neighbor ferromagnet $(\mathbb{Z}^{n-1}, H', \nu)$ in dimension $n-1$ with the same single-spin measure ν and coupling J:

$$H' = -J \sum_{i' \in \mathbb{Z}^{n-1}} \sum_{\alpha'=1}^{n-1} \sigma'_{i'} \sigma'_{i'+1_{\alpha'}} \tag{48}$$

Then for any inverse temperature β there exists an equilibrium state $\langle\ \rangle_{PS}$ of $(\mathbb{Z}^n, H, \nu\rangle$ such that

$$\begin{aligned} \langle \sigma_i \rangle_{PS} &\geqslant m_s & \forall i = (i_1,\dots,i_n) \in \mathbb{Z}^n:\quad i_1 \geqslant 0 \\ \langle \sigma_i \rangle_{PS} &\leqslant -m_s & \forall i = (i_1,\dots,i_n) \in \mathbb{Z}^n:\quad i_1 < 0 \end{aligned} \tag{49}$$

Since the spontaneous magnetization $m_s > 0$ if the reciprocal temperature β is sufficiently large, the state $\langle\ \rangle_{PS}$ has a sharp phase separation when β is large.

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